Tensor modes and the cosmic microwave background

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Introduction

Inflationary theories predict the existence of a stocastic background of gravitational waves (tensor perturbations to the metric). Detection of these waves would constrain inflationary models and increase our knowledge of the earliest moments of the Universe. While direct detection by interferometers such as LISA is a future possibility, indirect detection via the cosmic microwave background (CMB) is actively being pursued by groups like BICEP and QUAD.

Gravitational waves produce temperature perturbations via the integrated Sachs-Wolfe (ISW) effect. A photon propagating past a gravitational wave which is oscillating with changing amplitude acquires a net change in energy. Consequently an initially isotropic temperature distribution becomes anisotropic in the presence of gravitational waves. Compton scattering of this anisotropic temperature distribution generates polarisation.

Detailed numerical calculations of the statistical power spectra of the fluctuations exist. E.g. CMBFAST. Here we develop complementary analytic expressions that reproduce the main features of these calculations and help illustrate the underlying physics.

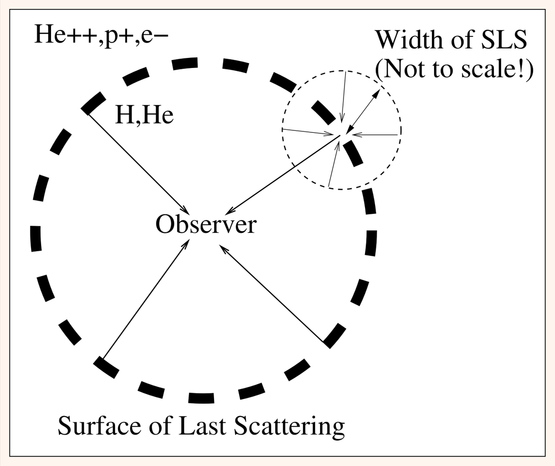

Fig. 1. The Cosmic Microwave Background (CMB) originates at the time of when protons and electrons combine to form neutral Hydrogen. Photons no longer scatter from charged particles and free-stream to the present day. Fluctuations at recombination are frozen in.

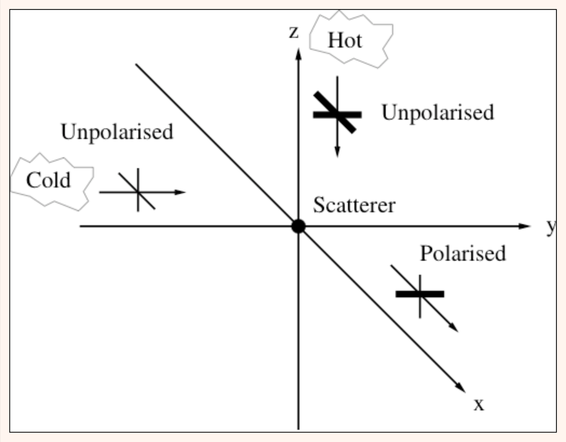

Fig. 2. Unpolarised light from an anisotropic temperature distribution Compton scatters from a charged particle leading to polarised outgoing light. The angular dependence of Compton scattering picks out the quadrupole part of the incident photon distribution.

Recombination History

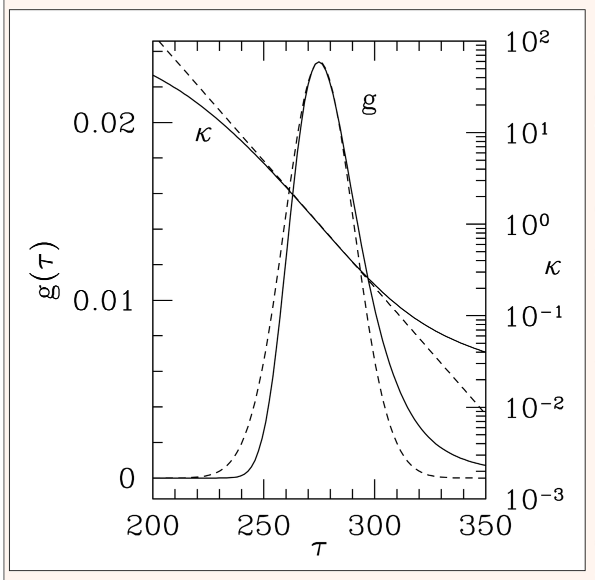

Fig. 4. Most relevant thermal history is contained in the visibility function $g(\tau)$ describing the probability that an observed photon last scattered at a conformal time τ. It is sharply peaked at the surface of last scattering and can be readily approximated by a narrow Gaussian. This approximation also reproduces the optical depth κ in the region of the SLS.

Formalism

To describe temperature and polarisation fluctuations projected onto the sky we utilise a harmonic decomposition of the fractional fluctuation e.g.

$$\delta T/T_{CMB} = \Delta_T(k,\tau,\theta,\phi) = \sum_l (2l+1)(-i)^l \Delta_{Tl}(k) P_l(\cos\theta)$$

The multipoles due to tensor modes are then given by the following line of sight integrals, $X=(E,B)$

$$\Delta_{Tl}(k) = \sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^{\tau_0} d\tau \left(-\dot{h}(\tau)e^{-\kappa} + g(\tau)\Psi(k,\tau)\right) P_{Tl}[k(\tau_0-\tau)],$$

$$\Delta_{Xl}(k) = \int_0^{\tau_0} d\tau \big(-g(\tau)\Psi(k,\tau)\big) P_{Xl}[k(\tau_0-\tau)].$$

The statistics of the fluctuations are then described by their power spectrum

$$C_{XX'l} = (4\pi)^2 \int k^2 dk P_h(k) \Delta_{Xl}(k) \Delta_{X'l}(k).$$

The sourcing term is given by

$$\Psi \equiv \left[\frac{1}{10}\tilde{\Delta}_{T0} + \frac{1}{7}\tilde{\Delta}_{T2} + \frac{3}{70}\tilde{\Delta}_{T4} - \frac{3}{5}\tilde{\Delta}_{P0} + \frac{6}{7}\tilde{\Delta}_{P2} - \frac{3}{70}\tilde{\Delta}_{P4}\right].$$

The equation of motion for the amplitude h of the two gravitational wave polarisation modes $(i=+,\times)$ is

$$\ddot{h}_i + 2\frac{\dot{a}}{a}\dot{h}_i + k^2 h_i = 16\pi G a^2 \Pi,$$

where Π is the tensor part of the anisotropic stress.

These expressions involve the following elements:

- Thermal history: visibility function $g(\tau)$ and optical depth κ.
- Harmonic projection: projection factors $P_{Xl}[k(\tau_0-\tau)]$.
- Initial conditions: primordial gravitational wave power spectrum $P_h(k)$.
- Gravitational wave evolution: $h(\tau)$.
- Growth of anisotropy: polarisation source term Ψ.

We approximate each of these elements in turn, finally arriving at an analytic expression for the polarisation power spectra

$$C_{Xl} = (4\pi)^2 \left(\frac{1}{7}\log\frac{10}{3}\right)^2 \int k^2 dk P_h(k) \langle P_{Xl}[k(\tau_0-\tau_R)]^2\rangle \dot{h}(\tau_R)^2 \Delta\tau_R^2\, e^{-(k\Delta\tau_R)^2}.$$

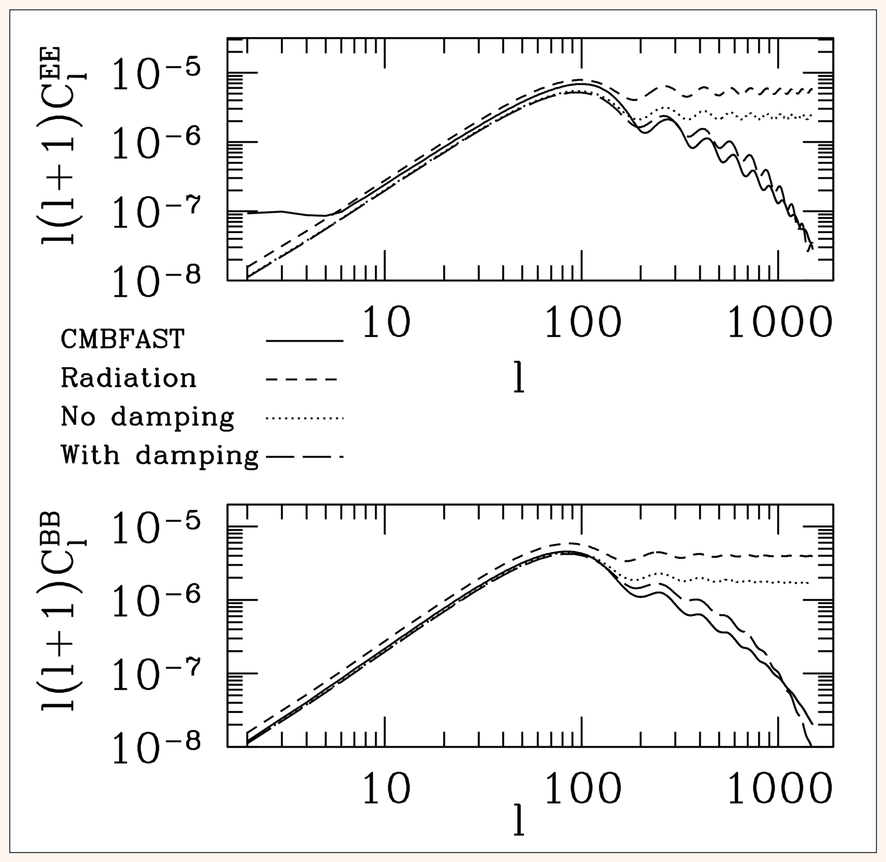

Fig. 3. Analytic approximations for the power spectrum reproduce the main features of numerical calculation using CMBFAST.

Projection

The functions for projecting from Fourier space to the unit sphere may be approximated by exploiting the narrowness of the visibility function to wri $\Delta_{Xl} \approx P_{Xl}[k(\tau_0-\tau_R)] \int_0^{\tau_0} d\tau\, g(\tau)\Psi(\tau).$

We then use Debye's asymptotic approximation for the Bessel function

$$j_l(x) = \frac{1}{\sqrt{x^2\sin\alpha}} \cos\left[x(\sin\alpha - \alpha\cos\alpha) - \pi/4\right]$$

where $\cos\alpha = (l+1/2)/x$.

and average to simplify the form for the projection factors which involve a sum of spherical Bessel functions.

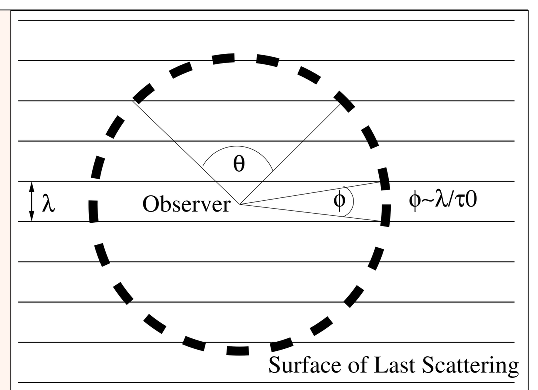

Fig. 5. A single k mode induces correlations on a range of angular scales. Peak contribution at $l \sim k(\tau_0-\tau_R)$.

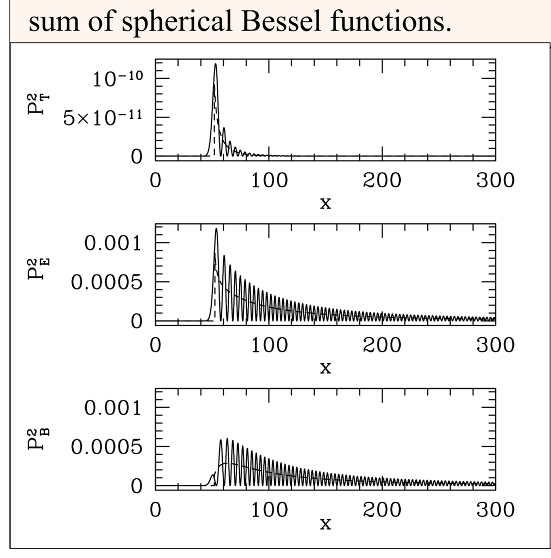

Fig. 6. The exact projection factors are highly oscillatory. Sharp peaks in P_T and P_E faithfully reproduce features imprinted in the fluctuations at recombination. The broad peak in P_B leads to smoothing of features in the observed power spectrum. Period averaged approximations are easier to calculate with. Plots are for l=50.

Gravitational Wave Evolution

Inflation predicts a nearly scale-invariant primordial gravitational wave power spectrum. This is processed by waves entering the cosmological horizon and evolving to generate the spectrum at recombination. Simple scaling arguments for the gravitational wave amplitude h then allow us to determine the scaling of the CMB power spectra

$$h \propto \begin{cases} 1, & k < 1/\tau_R, \\ k^{-2}, & 1/\tau_{\rm eq} > k > 1/\tau_R, \\ k^{-1}, & k > 1/\tau_{\rm eq}. \end{cases} \longrightarrow \quad l(l+1)C_l^{XX} \propto \begin{cases} l^2, & l < l_R, \\ l^{-2}, & l_{\rm eq} > l > l_R, \\ 1, & l_{\rm eq} < l < l_\Delta, \\ l^{-4}, & l > l_\Delta. \end{cases}$$

These regimes correspond to pre-horizon entry, matter dominated, radiation dominated and phase damped epochs.

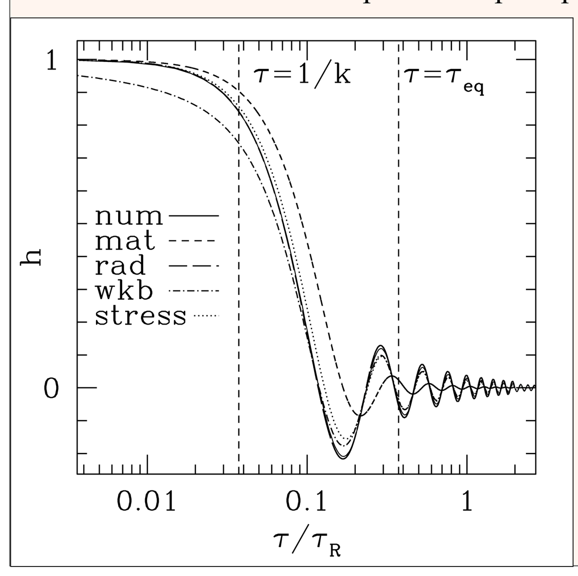

Fig. 7. Gravitational waves are frozen until horizon entry (kτ=1) after which they decay and oscillate. Evolution depends on the expansion rate and the presence of anisotropic stresses, which damp the amplitude. Exact analytic expressions for h exist in the matter and radiation dominated limit. In the mixed case the WKB approximation may be used.

Scalar and Tensor Power Spectra

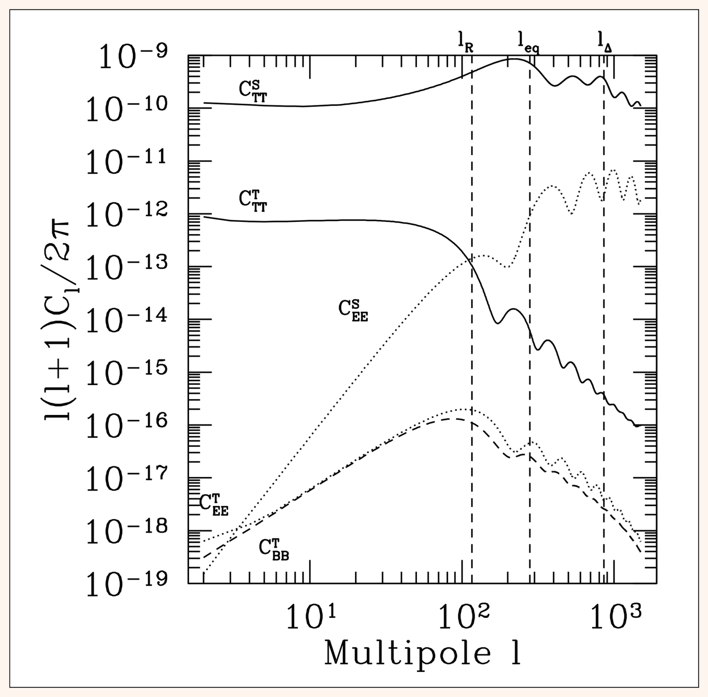

Fig. 10. CMB Power spectra describe the statistics of temperature and polarisation fluctuations. Features may be understood in terms of the horizon sizes at recombination l_R and matter-radiation l_{eq} and the width of the surface of last scattering l_Δ. Baryonic oscillation generates the features of the scalar spectrum. The oscillation of inflationary gravitational waves determines the tensor spectrum.

Growth of anisotropy

Before recombination the optical depth for photons is large leading to a strong coupling between photons and baryons. During this tightly-coupled period the short distance between scatterings suppresses the growth of anisotropy. As recombination proceeds the photon mean free path increases and anisotropy grows. This growth is described by the two Boltzmann equations

$$\dot{\tilde{\Delta}}_T + ik\mu\tilde{\Delta}_T = -\dot{h} - \dot{\kappa}[\tilde{\Delta}_T - \Psi],$$

$$\dot{\tilde{\Delta}}_P + ik\mu\tilde{\Delta}_P = -\dot{\kappa}[\tilde{\Delta}_P + \Psi].$$

These may be approximated by taking the optical depth to be large and expanding in inverse powers of κ. The resulting evolution equation for the source is

$$\dot{\Psi} + \frac{3}{10}\dot{\kappa}\Psi = -\frac{\dot{h}}{10}.$$

The evolution of Ψ is driven by the oscillation of the tensor mode. In solving this we exploit the narrowness of the visibility function which allows us to treat the value of $\dot{h}$ at the peak as representative. This works provided that we incorporate an exponential damping term due to phase damping on small scales. We arrive at an expression for the integrated source

$$\int_0^{\tau_0} d\tau\, g(\tau)\Psi(\tau) \approx \dot{h}(\tau_R)\Delta\tau_R e^{-(k\Delta\tau_R)^2/2} \left(\frac{1}{7}\log\frac{10}{3}\right)$$

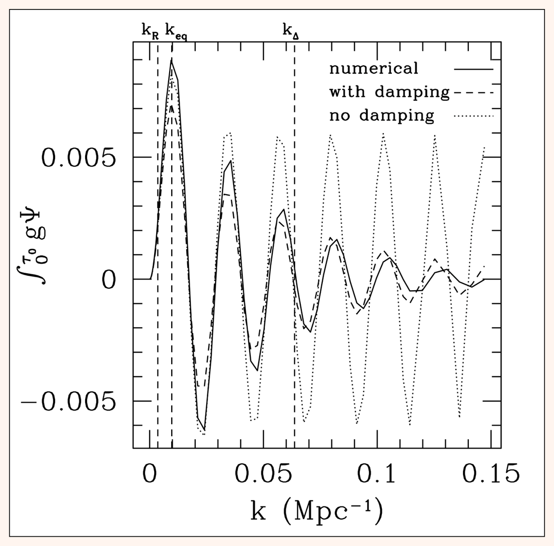

Fig. 8. Integrated source shows oscillation with wavenumber k. These features are projected into the angular power spectrum.

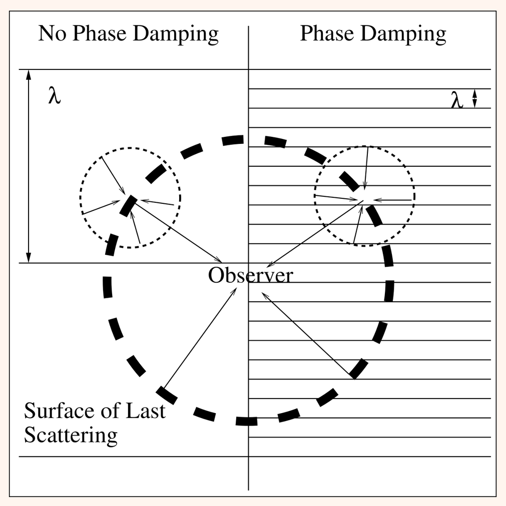

Fig. 9. On scales smaller than the width of the SLS coherent scattering of photons from different phase regions leads to cancellation and exponential suppression of the anisotropy.