

Bound States and Decay Times of Fermions in a Schwarzschild Black Hole Background

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Abstract

We compute the spectrum of normalized fermion bound states in a Schwarzschild black hole background. The eigenstates have complex energies. The real part of the energies, for small couplings, closely follow a Hydrogen-like spectrum. The imaginary contributions give decay times for the various states, due to the absorption properties of the hole. As expected, states closer to the hole have shorter half-lives. As the coupling increases, the spectrum departs from that of the Hydrogen atom, as states close to the horizon become unfavourable. Beyond a certain coupling the $1S_{1/2}$ state is no longer the ground state, which shifts to the $2P_{3/2}$ state. For each positive energy state a negative energy counterpart exists, with opposite sign of its real energy, and the same decay factor. It follows that the Dirac sea of negative energy states is decaying, which may provide a physical contribution to Hawking radiation.

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1 Introduction

Quantum theory in a black hole background has been extensively studied by many authors. Detailed discussions of this problem are contained in the books by Birrell & Davies [1] and Chandrasekhar [2], and the review paper by Brout *et al.* [3]. Much of the attention in this work is focussed on the wave equation and its scattering properties. Detailed studies of the Dirac equation in a black hole background are less common. Indeed, the lowest order scattering cross section for a fermion in a black hole background has only recently been computed [4, 5]. In this paper we investigate another previously neglected aspect of quantum mechanics in a black hole background. This is the existence of the bound state spectrum for particles orbiting a spherically-symmetric point source. That is, we study the gravitational analogue of the Hydrogen atom orbitals.

There has been strangely little effort devoted to the study of the bound state spectrum, despite the fundamental importance of the electromagnetic analogue. But it is clear that these states must exist — how else can one provide a quantum description of a particle in orbit around a black hole? These states must also be essential in the quantum description of the capture process. The problem was discussed in 1974 by Deruelle and Ruffini [6], who described the existence of resonance states in the Klein-Gordon equation. Further significant contributions were made in a series of papers by Gaina and coauthors [7, 8, 9]. These papers

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give various analytic expressions for the real and imaginary parts of the energy in a series of limiting cases. But to our knowledge the complete energy level diagrams presented here have not been previously computed.

For a particle of mass m in the field of a black hole of mass M the dimensionless coupling strength is defined by

$$\alpha = \frac{mM}{m_p^2} \quad (1)$$

where m_p is the Planck mass. In this paper we compute the fermion bound state spectra for α in the range $0 \cdots 1$. If the bound particle is assumed to be an electron, this range corresponds to black holes of masses up to 5×10^{14} kg, which is the scale appropriate for primordial black holes formed in the early universe. Computing the energy spectrum is more complicated than the Hydrogen atom case for a range of reasons. The first is that the radially-separated Dirac equation contains three singular points, only two of which are regular. There is no special function theory appropriate for the study of such equations, so we have to resort to a range of numerical techniques to find the spectrum. The second problem is that the Hamiltonian for a Schwarzschild black hole is not Hermitian. This is due to the presence of the singularity, which acts a current sink. It follows that we must search for eigenstates over the two-dimensional space of complex energies.

Despite these difficulties, the problem can be tackled numerically, and we present a range of results for the real and imaginary parts of the energy. These are sufficient to predict how the spectrum will behave for larger values of the coupling constant. The first result, which is entirely to be expected, is that the orbitals become increasingly tightly bound as the coupling increases. It follows that, for a given state, the energy will initially decrease with α , but will eventually turn round and start increasing as the particle spends too much time inside the classical radius of minimum energy. States with higher angular momentum then become energetically favourable as α increases. In particular, we show that beyond $\alpha \approx 0.6$ the $1S_{1/2}$ state is no longer the ground state. While the real part of the energy behaves in quite a complicated fashion, the imaginary part, which controls the decay rate, simply increases in magnitude. This is also as one would expect. The closer the orbital density is to the singularity, the greater the probability of capture.

We start by discussing the Dirac equation in a Schwarzschild background in an arbitrary gauge. This is helpful in establishing a range of gauge-invariant results. In particular, the energy conjugate to time translation symmetry is confirmed to be a gauge invariant quantity. This is important in order to guarantee that the quantity is a physical observable. We next establish the behaviour of the wavefunction around the horizon, which is sufficient to establish that the states must decay exponentially with time. We then turn to a specific choice of gauge which is well-suited to numerical solution. We solve the equations by simultaneously integrating out from the horizon and in from infinity. We then vary the energy to ensure that the solutions match at some finite radius. This process guarantees that we find a global, normalizable bound state. A set of spectra are obtained, and the density is plotted for a range of states. Decay rates and expectation values for the distance are also presented. We end with a discussion of the significance of these bound states, and the possible physical

processes that they may generate. Except where stated otherwise, natural units with $G = \hbar = c = 1$ are assumed throughout. We employ a spacetime metric with signature -2 .

2 The Dirac equation

We start by defining a general parameterisation of the Schwarzschild solution. This general form will help to guarantee that various expressions are gauge invariant. We let $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ denote the standard gamma matrices in the Dirac–Pauli representation, and introduce polar coordinates (r, θ, ϕ) . From these we define the unit polar matrices

$$\begin{aligned}\gamma_r &= \sin\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) + \cos\theta\gamma_3 \\ \gamma_\theta &= \cos\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) - \sin\theta\gamma_3 \\ \gamma_\phi &= -\sin\phi\gamma_1 + \cos\phi\gamma_2.\end{aligned}\tag{2}$$

In terms of these we define the four matrices

$$\begin{aligned}g^t &= a_1\gamma_0 - a_2\gamma_r & g^\theta &= -\frac{1}{r}\gamma_\theta \\ g^r &= -b_1\gamma_r + b_2\gamma_0 & g^\phi &= -\frac{1}{r\sin\theta}\gamma_\phi\end{aligned}\tag{3}$$

where (a_1, a_2, b_1, b_2) are scalar functions of r satisfying

$$\begin{aligned}a_1b_1 - a_2b_2 &= 1 \\ (b_1)^2 - (b_2)^2 &= 1 - 2M/r.\end{aligned}\tag{4}$$

The reciprocal set of matrices are therefore

$$\begin{aligned}g_t &= b_1\gamma_0 - b_2\gamma_r & g_\theta &= r\gamma_\theta \\ g_r &= a_1\gamma_r - a_2\gamma_0 & g_\phi &= r\sin\theta\gamma_\phi.\end{aligned}\tag{5}$$

These matrices satisfy

$$\begin{aligned}\{g^\mu, g^\nu\} &= 2g^{\mu\nu}I \\ \{g_\mu, g_\nu\} &= 2g_{\mu\nu}I \\ \{g^\mu, g_\nu\} &= 2\delta_\nu^\mu I\end{aligned}\tag{6}$$

where μ runs over the set (t, r, θ, ϕ) , I is the identity matrix, and $g_{\mu\nu}$ is the spacetime metric. The line element defined by this metric is

$$\begin{aligned}g_{\mu\nu}dx^\mu dx^\nu &= (1 - 2M/r)dt^2 + 2(a_1b_2 - a_2b_1)dt dr - ((a_1)^2 - (a_2)^2)dr^2 \\ &\quad - r^2(d\theta^2 + \sin^2\theta d\phi^2).\end{aligned}\tag{7}$$

This line element is the most general form one can adopt for the Schwarzschild solution. There is only one degree of freedom in equation (7), since the terms are related by

$$(1 - 2M/r)((a_1)^2 - (a_2)^2) + (a_1b_2 - a_2b_1)^2 = 1.\tag{8}$$

This arbitrary degree of freedom corresponds to the fact that the time coordinate is only defined up to an arbitrary radially-dependent term. That is, we can set

$$\bar{t} = t + \alpha(r), \quad (9)$$

and the new line element will be independent of the new time coordinate \bar{t} . Rather than think in terms of changing the time coordinate, however, it is simpler for our purposes to always label the time coordinate as t and instead redefine a_1 and a_2 . These then transform as

$$\begin{aligned} a_1 &\mapsto \bar{a}_1 = a_1 - b_2 \alpha' \\ a_2 &\mapsto \bar{a}_2 = a_2 - b_1 \alpha', \end{aligned} \quad (10)$$

with b_1 and b_2 unchanged. Throughout dashes denote derivatives with respect to r . It is straightforward to confirm that the new set $(\bar{a}_1, \bar{a}_2, b_1, b_2)$ still satisfy the constraints of equation (4).

The four variables $(a_1 \dots b_2)$ are subject to two constraint equations, so must contain two arbitrary degrees of freedom. The first arises from the freedom in the time coordinate as described in equation (10). The second lies in the freedom to perform a radially-dependent boost, which transforms the variables according to

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \mapsto \begin{pmatrix} \cosh \beta & \sinh \beta \\ \sinh \beta & \cosh \beta \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \quad (11)$$

where β is an arbitrary, non-singular function of r . This boost does not alter the line element of equation (7). Outside the horizon we have $|b_1| > |b_2|$, and in the asymptotically flat region b_1 can be brought to +1 by a suitable boost. It follows that we must have

$$b_1 > 0 \quad \forall r \geq 2M. \quad (12)$$

At the horizon we therefore have b_1 positive, and $b_2 = \pm b_1$. For black holes (as opposed to white holes) the negative sign is the correct one, as this choice guarantees that all particles fall in across the horizon in a finite proper time. This sign is also uniquely picked out by models in which the black hole is formed by a collapse process. We can therefore write

$$b_2 = -b_1 \quad \text{at } r = 2M. \quad (13)$$

Combining this with the identity $a_1 b_1 - a_2 b_2 = 1$ we find that, at the horizon, we must have

$$a_1 b_2 - a_2 b_1 = -1 \quad \text{at } r = 2M. \quad (14)$$

The diagonal form of the Schwarzschild metric sets $a_1 b_2 - a_2 b_1 = 0$, so does not satisfy this criteria. But for this case the time coordinate t is only defined outside the horizon, and the horizon itself is not dealt with correctly.

We now have a general parameterisation of the Schwarzschild solution in an arbitrary gauge. The next step is to write down the Dirac equation in this background. This is

$$i g^\mu \nabla_\mu \psi = m \psi, \quad (15)$$

where

$$\nabla_\mu \psi = \left(\partial_\mu + \frac{i}{2} \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta} \right) \psi, \quad \Sigma_{\alpha\beta} = \frac{i}{4} [\gamma_\alpha, \gamma_\beta]. \quad (16)$$

The components of the spin connection are found in the standard way (see Nakahara [10], for example). These turn out to give

$$g^\mu \frac{i}{2} \Gamma_\mu^{\alpha\beta} \Sigma_{\alpha\beta} = \left(b'_2 + \frac{2b_2}{r} \right) \gamma_0 - \left(b'_1 + \frac{2(b_1 - 1)}{r} \right) \gamma_r. \quad (17)$$

For the Dirac spinor we use a radial separation of the form

$$\psi = \frac{e^{-iEt}}{r} \begin{pmatrix} u_1(r) \chi_\kappa^\mu(\theta, \phi) \\ u_2(r) \sigma_r \chi_\kappa^\mu(\theta, \phi) \end{pmatrix} \quad (18)$$

where E is the (complex) energy and

$$\sigma_r = \sin\theta(\cos\phi \sigma_1 + \sin\phi \sigma_2) + \cos\theta \sigma_3. \quad (19)$$

The angular eigenmodes are labeled by κ , which is a positive or negative nonzero integer, and μ , which is the total angular momentum in the $\theta = 0$ direction. Our convention for these eigenmodes is that

$$(\boldsymbol{\sigma} \cdot \mathbf{L} + \hbar) \chi_\kappa^\mu = \kappa \hbar \chi_\kappa^\mu, \quad \kappa = \dots, -2, -1, 1, 2, \dots \quad (20)$$

The positive and negative κ modes are related by

$$\sigma_r \chi_\kappa^\mu = \chi_{-\kappa}^\mu. \quad (21)$$

The trial function (18) results in the pair of coupled first-order equations

$$\begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \mathbf{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (22)$$

where

$$\mathbf{B} = \begin{pmatrix} \kappa/r - b'_1/2 + ia_2E & i(m + a_1E) - b'_2/2 \\ -i(m - a_1E) - b'_2/2 & -\kappa/r - b'_1/2 + ia_2E \end{pmatrix}. \quad (23)$$

These are the equations we wish to solve for complex energy E . It is first worthwhile confirming that the equations are gauge invariant. A redefinition of the time coordinate is equivalent to the transformations described in equation (10). These are combined with the transformation

$$u_1 \mapsto u_1 e^{-iE\alpha} \quad u_2 \mapsto u_2 e^{-iE\alpha} \quad (24)$$

which together ensure that equation (22) is still satisfied. The radial boost defined by equation (11) is combined with the transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \cosh(\beta/2) & -\sinh(\beta/2) \\ -\sinh(\beta/2) & \cosh(\beta/2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (25)$$

to again ensure that the equation is still satisfied. In either case we see that the eigenvalue E is unchanged, so is a true gauge-invariant quantity.

The angular separation of equation (18) is clearly justified from the form of the Dirac equation. The separation into energy eigenstates is gauge invariant, but it is helpful to see the separation in a gauge where the Dirac equation takes on a manifestly Hamiltonian form. This is provided by the ‘Newtonian’ gauge [4, 11] which sets

$$\begin{aligned} a_1 &= 1 & a_2 &= 0 \\ b_1 &= 1 & b_2 &= -(2M/r)^{1/2}. \end{aligned} \quad (26)$$

In this gauge the Dirac equation takes on the simple form

$$i\rlap{\not{\partial}}\psi - i\gamma^0 \left(\frac{2M}{r}\right)^{1/2} \left(\frac{\partial}{\partial r} + \frac{3}{4r}\right)\psi = m\psi, \quad (27)$$

where $\rlap{\not{\partial}}$ is the Dirac operator in flat Minkowski spacetime. This equation is manifestly separable in time, so has solutions which go as $\exp(-iEt)$. Since the separation works in this gauge, it must work in all others. We will return to this gauge choice when we turn to finding the energy spectrum.

The nature of equation (22) can be understood more clearly by writing it in the form

$$(1 - 2M/r) \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} b_1 & -b_2 \\ -b_2 & b_1 \end{pmatrix} \mathbf{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (28)$$

This exposes the fact that the horizon is a regular singular point of the radial equations. The same is true of the origin, and infinity turns out to be an irregular singular point. This implies that the radial equations cannot be manipulated into second-order hypergeometric form, as is the case for the Hydrogen atom. The closest the equations come to a recognisable form is that of Heun’s equation, which generalises the hypergeometric equation to the case of four regular singular points on the complex plane [12]. But Heun’s equation can usually only be analysed using numerical techniques, and these are the tools we will apply to equation (22)

The presence of singular points means that we must check carefully that our solutions behave appropriately at these points. The point at infinity is not an issue, as we seek solutions which decay exponentially. Similarly, the origin is not a problem. We expect that the function will be weakly singular there as the origin acts as a current sink, and this is indeed the case. The horizon, however, is more complicated. The wavefunction must be well behaved at the horizon if it is to represent a physical solution. To test this we introduce the series expansion

$$u_1 = \eta^s \sum_{k=0}^{\infty} \alpha_k \eta^k, \quad u_2 = \eta^s \sum_{k=0}^{\infty} \beta_k \eta^k, \quad (29)$$

where $\eta = r - 2M$. On substituting this series into equation (28), and setting $\eta = 0$, we obtain the indicial equation

$$\det \left[\begin{pmatrix} b_1 & -b_2 \\ -b_2 & b_1 \end{pmatrix} \mathbf{B} - \frac{s}{r} \mathbf{I} \right]_{r=2M} = 0, \quad (30)$$

where I is the identity matrix. Employing the result that

$$b_1 b'_1 - b_2 b'_2 = M/r^2 \quad (31)$$

we find that the two solutions of the indicial equation are

$$s = 0, -\frac{1}{2} + 4iME(b_1 a_2 - b_2 a_1)_{r=2M}. \quad (32)$$

Equation (14) then tells us that the two indices are

$$s = 0, -\frac{1}{2} + 4iME. \quad (33)$$

These indices are therefore gauge invariant. The regular root $s = 0$ ensures that we can always construct a solution which is finite and continuous at the horizon. The singular branch gives rise to discontinuous solutions with an outgoing current at the horizon. These can be used to provide a heuristic explanation of the Hawking radiation [11]. It is clear that the non-zero indicial root gives rise to a wavefunction which is ill-defined at the horizon, and so cannot represent a physical state. We must therefore confine our search for bound states to solutions which are regular at the horizon.

Eigenmodes with different values of κ and μ are orthogonal. For states with the same values of κ and μ the quantum inner product is

$$\langle \psi | \phi \rangle = \int_0^\infty dr (a_1(u_1^* v_1 + u_2^* v_2) + a_2(u_2^* v_1 + u_1^* v_2)), \quad (34)$$

where the u_i and v_i denote the radial functions in ψ and ϕ respectively. Current conservation for the Dirac equation is summarised in the relation

$$\begin{aligned} \frac{\partial}{\partial t} (a_1(u_1 u_1^* + u_2 u_2^*) + a_2(u_1 u_2^* + u_2 u_1^*) e^{-i(E-E^*)t}) \\ = -\frac{\partial}{\partial r} (b_1(u_1 u_2^* + u_2 u_1^*) + b_2(u_1 u_1^* + u_2 u_2^*) e^{-i(E-E^*)t}). \end{aligned} \quad (35)$$

Again it is straightforward to confirm that this equation is gauge invariant. The right-hand side of this equation defines r^2 times the radial flux. We denote this by J ,

$$J(r) = b_1(u_1 u_2^* + u_2 u_1^*) + b_2(u_1 u_1^* + u_2 u_2^*). \quad (36)$$

For normalizable states we must have $J \mapsto 0$ as $r \mapsto \infty$. But at the horizon we also have

$$J = -b_1 |u_1 - u_2|^2, \quad (37)$$

which defines an inward-pointing current. At the horizon, the regular solution has

$$|u_1 - u_2|^2 = |\alpha_0 - \beta_0|^2, \quad (38)$$

using the power series expansion of equation (29). The coefficients are related by

$$\left(iE - \frac{1}{8M} + b_1 \left(\frac{\kappa}{2M} - im \right) \right) \alpha_0 = \left(-iE + \frac{1}{8M} + b_1 \left(\frac{\kappa}{2M} - im \right) \right) \beta_0. \quad (39)$$

It is therefore impossible to satisfy $\alpha_0 = \beta_0$ for finite energy, so there must be a non-vanishing inward current present at the horizon. This in turn tells us that the state must decay. This decay takes place at the origin, where Hermiticity breaks down [11, 13]. For bound states the energy E must contain real and imaginary terms, so we set

$$E = \omega - i\nu. \quad (40)$$

Current conservation now takes the form

$$\frac{dJ}{dr} = 2\nu(a_1(u_1u_1^* + u_2u_2^*) + a_2(u_1u_2^* + u_2u_1^*)). \quad (41)$$

Given a set (u_1, u_2, E, κ) which solve the radial equation (22) a new solution set is generated by the transformation

$$(u_1, u_2, E, \kappa) \mapsto (u_2^*, u_1^*, -E^*, -\kappa). \quad (42)$$

It follows that the real part of the energy spectrum is symmetric about the zero point. That is, for a state with real energy ω a corresponding antiparticle state exists with real energy $-\omega$. The decay rate is the same for both states, however. If we assume that the vacuum is constructed from the Dirac sea of negative energy states, then this vacuum will decay in time. A loss of negative energy states can be equally interpreted as generation of positive energy states, which provides a suggestive physical model for Hawking radiation.

3 The energy spectrum

To solve for the energy spectrum we work mainly in the Newtonian gauge of equation (26). In this gauge the interaction with the black hole is defined solely by the interaction Hamiltonian

$$H_I\psi = i\hbar \left(\frac{2GM}{r}\right)^{1/2} \frac{1}{r^{3/4}} \frac{\partial}{\partial r} (r^{3/4}\psi). \quad (43)$$

Dimensional constants are included in a number of equations in this section to illustrate certain features of the problem. The conserved inner product between states in this gauge has the usual flat-space form

$$\langle \psi | \phi \rangle = \int d^3x \psi^\dagger \phi. \quad (44)$$

The interaction Hamiltonian is not Hermitian, as we have

$$H_I - H_I^\dagger = -i\hbar(2GM/r^3)^{1/2}\delta(\mathbf{x}). \quad (45)$$

It is straightforward to check that all wavefunctions approach the origin as $r^{-3/4}$, so the non-Hermitian part of H_I has finite expectation. This confirms that Hermiticity only breaks down at the origin, as stated earlier. Furthermore, if the state is normalized such that

$$\int dr (u_1u_1^* + u_2u_2^*) = 1 \quad (46)$$

then the imaginary component of the energy, $-i\nu$, is determined by

$$\nu = \lim_{r \rightarrow 0} \frac{\hbar(2GM)^{1/2}}{2} \frac{1}{r^{3/2}} (u_1 u_1^* + u_2 u_2^*). \quad (47)$$

This identity only holds if the state is globally normalized. It provides a further independent check that the solutions we obtain numerically are globally normalizable bound states.

The interaction Hamiltonian is independent of the speed of light, so the non-relativistic approximation to the Dirac equation results in the Schrödinger equation

$$-\frac{\hbar^2 \nabla^2}{2m} \psi_{\text{NR}} + i\hbar \left(\frac{2GM}{r} \right)^{1/2} \frac{1}{r^{3/4}} \frac{\partial}{\partial r} (r^{3/4} \psi_{\text{NR}}) = E_{\text{NR}} \psi_{\text{NR}}, \quad (48)$$

where the subscript NR denotes non-relativistic. If we now introduce the phase-transformed variable

$$\Psi = \psi_{\text{NR}} \exp\left(-i(8r/a_0)^{1/2}\right) \quad (49)$$

where

$$a_0 = \frac{\hbar^2}{GMm^2} \quad (50)$$

we see that Ψ satisfies

$$-\frac{\hbar^2 \nabla^2}{2m} \Psi - \frac{GMm}{r} \Psi = E_{\text{NR}} \Psi. \quad (51)$$

In the non-relativistic limit the energy spectrum is therefore given by the gravitational analogue of the Hydrogen atom spectrum [8],

$$E_{\text{NR}} = -\frac{G^2 M^2 m^3}{2\hbar^2} \frac{1}{n^2}, \quad n = 1, 2, \dots \quad (52)$$

In terms of the Planck mass m_p we can also write

$$E_{\text{NR}} = -\left(\frac{Mm}{m_p^2}\right)^2 \frac{mc^2}{2n^2}. \quad (53)$$

The fact that we have a reasonable starting point for the spectrum in the weak-coupling limit is valuable, as our method involves searching for eigenvalues over the complex energy plane. By analogy with the Hydrogen atom case, we expect that the non-relativistic spectrum will be a reasonable approximation provided

$$\frac{Mm}{m_p^2} \ll 1. \quad (54)$$

Returning to the full, relativistic equation (22), we convert this to dimensionless form by introducing the dimensionless distance variable

$$x = \frac{rc^2}{GM}, \quad (55)$$

which ensures that the horizon lies at $x = 2$. We also introduce the dimensionless coupling coefficient

$$\alpha = \frac{Mm}{m_p^2} \quad (56)$$

and energy

$$\varepsilon = \frac{EM}{c^2 m_p^2}. \quad (57)$$

In terms of these our eigenvalue problem becomes

$$(x-2) \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1 & (2/x)^{1/2} \\ (2/x)^{1/2} & 1 \end{pmatrix} \cdot \begin{pmatrix} \kappa & ix(\alpha + \varepsilon) - (8x)^{-1/2} \\ -ix(\alpha - \varepsilon) - (8x)^{-1/2} & -\kappa \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (58)$$

where the dashes now denote derivatives with respect to x . We seek eigenvalues ε for fixed coupling α .

Two complimentary methods are employed to solve the eigenvalue problem. We start with a series expansion around the horizon of the regular branch of the solution. The restriction to this branch removes two degrees of freedom at the horizon, so the function is uniquely specified up to an overall magnitude and phase. These are chosen conveniently by setting $u_1 = 1$ at the horizon. The power series expansion extends the solution a short distance away from the horizon, from where the values of (u_1, u_2) can be used to initiate numerical integration of the differential equation (58). For most values of ε the numerical integrator will start to increase exponentially after a finite distance. The aim initially is to vary ε so as to push this distance out as far as possible. This requires a reasonable initial guess for the eigenvalues, which is where the non-relativistic approximation is helpful to get things started.

Once we have achieved a reasonably accurate value for ε , we turn to a more sophisticated method to improve accuracy. We seek normalizable states for which ψ is finite over all space. To be confident we have found such a state we need to numerically integrate inwards from infinity, as well as outwards from the horizon. If the solutions for u_1 and u_2 can be arranged to match at some suitable radius then we have found a global solution to the first-order equations (58). To expand about infinity we need to take care of the essential singularity present there. A suitable series expansion is provided by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \exp\left(-px + 2i\varepsilon(2x)^{1/2} + \frac{\alpha^2 - 2p^2}{p} \ln x\right) \sum_{n=0} \begin{pmatrix} \alpha_{n/2} x^{-n/2} \\ \beta_{n/2} x^{-n/2} \end{pmatrix} \quad (59)$$

where

$$p^2 = \alpha^2 - \varepsilon^2 = \frac{M^2}{m_p^4 c^4} (m^2 c^4 - E^2). \quad (60)$$

The definition of p involves a complex square root, and the branch is chosen so that p has a positive real value, ensuring the wavefunction falls off exponentially.

The fact that only one root of the indicial equation is used implies that, for a given ε , ψ is specified at infinity up to an arbitrary magnitude and phase. The first few terms in the series expansion (59) are used to compute ψ at a finite radius and these values are then numerically integrated inwards. A certain amount of fine tuning is then required to pick the radius at which to attempt matching. Once a radius is chosen the matching condition is that the inward and outward values of the two complex functions u_1 and u_2 agree. This condition is converted into a set of four scalar equations which state that the real and imaginary differences vanish. In addition we have four arbitrary parameters to vary — the real and imaginary terms in the energy, and the magnitude and phase of the function integrated inwards from infinity. This system of four equations and four unknowns is then solved by a Newton–Raphson method. This converges very quickly and affords good control over accuracy.

Three independent checks were performed on the energy spectrum achieved by this method. The first was that the calculations were repeated using the same scheme in a different gauge. The gauge chosen for comparison is defined by advanced Eddington–Finkelstein coordinates, with

$$\begin{aligned} a_1 &= 1 + M/r & a_2 &= M/r \\ b_1 &= 1 - M/r & b_2 &= -M/r. \end{aligned} \tag{61}$$

The second test involved using a minimax routine to find the energy spectrum. This method is less accurate, but gave good agreement for the states of lowest energy. The final check was to confirm that, after normalisation, the states satisfy the identity of equation (47). This check was again satisfied to high precision.

4 Results

The real parts of the energy for the three lowest-energy states are plotted in figure 1. The vertical axis plots the real part of

$$\frac{\varepsilon}{\alpha} = \frac{E}{mc^2}. \tag{62}$$

This returns the energy in units of the rest energy of the particle. The fact that we obtain this dimensionless ratio reflects the equivalence principle. The mass m does not effect the spectrum on its own — the spectrum only depends on the product mM . States are labelled using the standard spectroscopic scheme. In this scheme $\kappa = 1$ corresponds to $S_{1/2}$, $\kappa = 2$ to $P_{3/2}$ and $\kappa = -1$ to $P_{1/2}$. For each eigenvalue κ a ladder of levels is obtained.

The energy spectrum illustrates a number of remarkable features. For small α the spectrum resembles that of a Hydrogen atom. But as the coupling increases the energy of the $1S_{1/2}$ state reaches a minimum and then starts to increase. The gravitational case avoids the $Z = 137$ catastrophe of the relativistic Coulomb problem. This is to be expected — coupling strengths with $\alpha > 1$ are routinely achieved astrophysically and such objects appear to be stable. We also see that as α increased beyond 0.6 the $P_{3/2}$ state appears to take over as the ground state. This is confirmed in figure 2, which just shows the spectra of the $S_{1/2}$ and $P_{3/2}$ states. We see clearly that around $\alpha = 0.65$ the $2P_{3/2}$ state takes

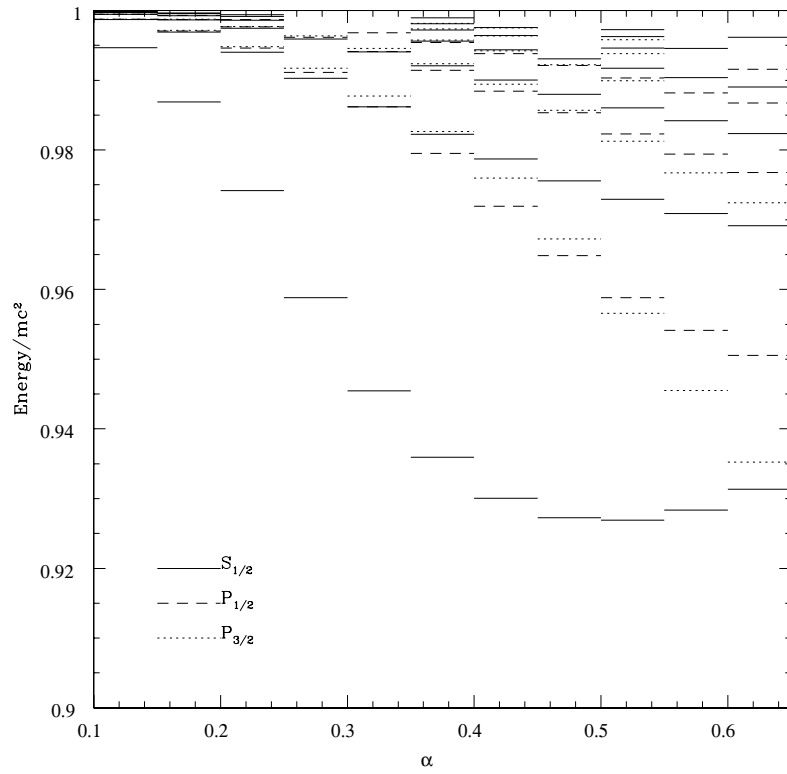


Figure 1: The real part of the bound state energy, in units of mc^2 . The horizontal axis labels the dimensionless coupling coefficient α , and the lines represent the value of the energy for the coupling at the left of the line, with α ranging from 0.1 to 0.6 in steps of 0.05. The $S_{1/2}$, $P_{1/2}$ and $P_{3/2}$ orbits are shown.

over from $1S_{1/2}$ as the ground state. An explanation of this phenomena can be found in the classical expression for the binding energy in a Schwarzschild potential. For a particle of mass m in a circular orbit at radius r the conserved relativistic energy, conjugate to time translation, is

$$E = mc^2 \frac{r - 2GM/c^2}{r^{1/2}(r - 3GM/c^2)^{1/2}}. \quad (63)$$

The radius r and angular momentum L are related by

$$\frac{L^2}{m^2} = \frac{GM r^2}{r - 3GM/c^2}. \quad (64)$$

Now suppose we attempt a form of naive Bohr quantisation by setting

$$L = n\hbar. \quad (65)$$

Converting to dimensionless quantities we find that, classically,

$$\frac{\varepsilon}{\alpha} = \frac{x - 2}{(x(x - 3))^{1/2}} \quad (66)$$

where

$$x = \frac{n^2}{2\alpha^2} \left(1 + \left(1 - \frac{12\alpha^2}{n^2} \right)^{1/2} \right). \quad (67)$$

In the small α regime this reproduces the spectrum of equation (53). But as α increases the energy falls to a minimum at $\alpha^2 = n^2/12$, beyond which the orbit no longer exists for a given n . The minimum energy achieved is $0.94mc^2$, corresponding to $x = 6$. Inside this radius no stable classical circular orbits exist. In the quantum description we find that as α increases the orbits get more tightly bound around the horizon. As the coupling increases the orbits are dominated by terms inside $x = 6$ and so become energetically less favourable. The ground state is then one of higher angular momentum, for which the orbit is less tightly bound.

The radial form of the wavefunction is best visualised by plotting r^2 times the timelike component of the current. We denote this ρ , and with our present gauge choices we have

$$\rho = |u_1|^2 + |u_2|^2. \quad (68)$$

The gauge invariant definition of ρ is that it is r^2 times the density as measured by observers in radial free-fall from rest at infinity. The first four $S_{1/2}$ states for small coupling are shown in figure 3. The plots are very similar to those for the non-relativistic hydrogen atom. In all cases the peak of the wavefunction is a long way outside the horizon, with only a small fraction of the probability density lying inside the horizon.

As we increase the coupling to $\alpha = 0.35$ we obtain the series of plots in figure 4. Predictably, the wavefunctions start to bunch in closer to the horizon. Slightly more surprisingly, the nodal structure disappears for larger couplings. The density no longer drops down to near zero at a number of nodes, but instead a number of dips are present. If we increase the coupling further still, to

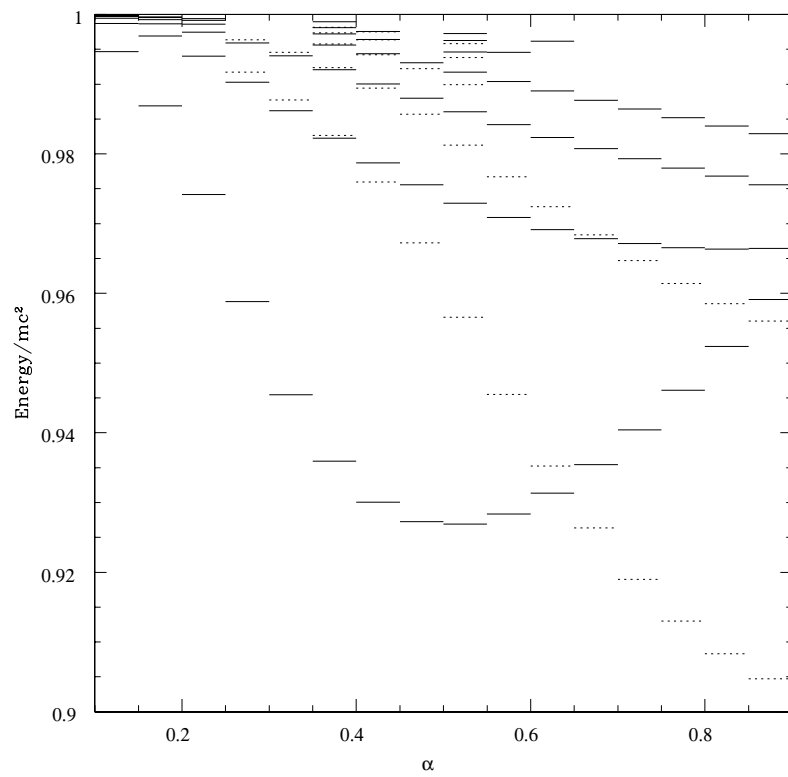


Figure 2: The energy spectra of the $S_{1/2}$ (solid) and $P_{1/2}$ (dashed) states. Beyond a coupling of around $\alpha = 0.65$ the $2P_{3/2}$ state takes over as the system's ground state.

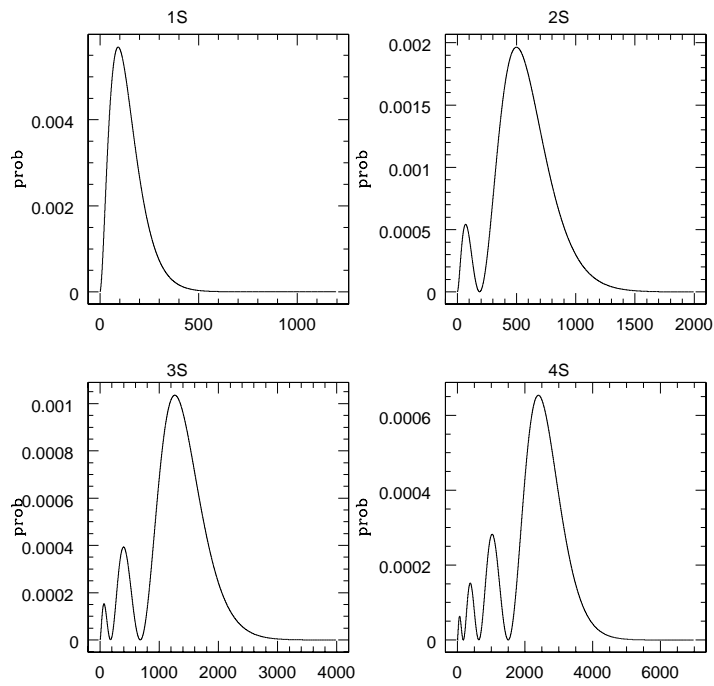


Figure 3: The radial probability density for the $1S_{1/2}$, $2S_{1/2}$, $3S_{1/2}$ and $4S_{1/2}$ states for a coupling of $\alpha = 0.1$. The horizontal axis is the dimensionless radius x , and the horizon lies at $x = 2$. All plots are started from the horizon. The part of the density inside the horizon is not plotted, though in all cases this smoothly approaches the origin.

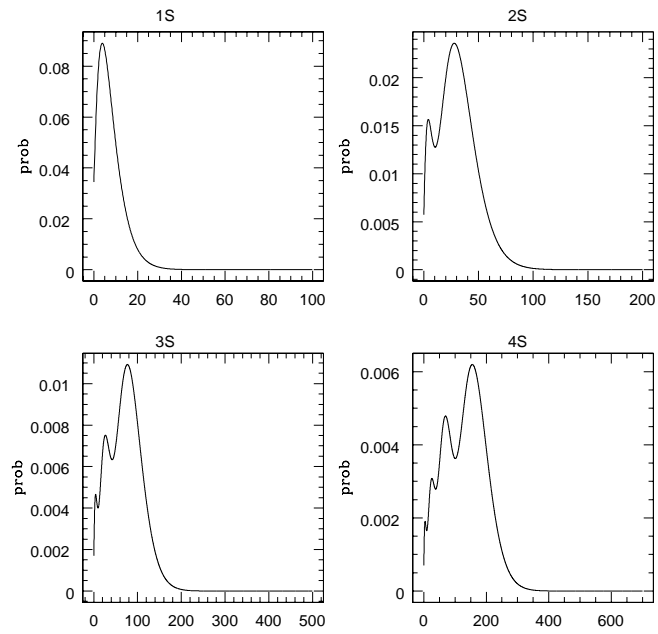


Figure 4: The radial probability density for the $1S_{1/2}$, $2S_{1/2}$, $3S_{1/2}$ and $4S_{1/2}$ states for a coupling of $\alpha = 0.35$. The nodal pattern seen in figure 3 is beginning to get washed out as the wavefunction compresses around the horizon.

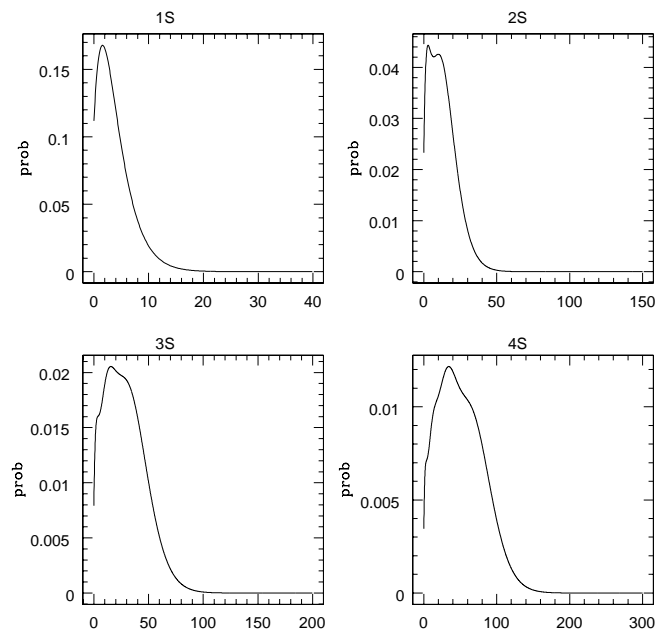


Figure 5: The radial probability density for the $1S_{1/2}$, $2S_{1/2}$, $3S_{1/2}$ and $4S_{1/2}$ states for a coupling of $\alpha = 0.5$. The pattern of nodes and dips seen in figures 3 and 4 has almost completely vanished, leaving a series of rather structureless density profiles.

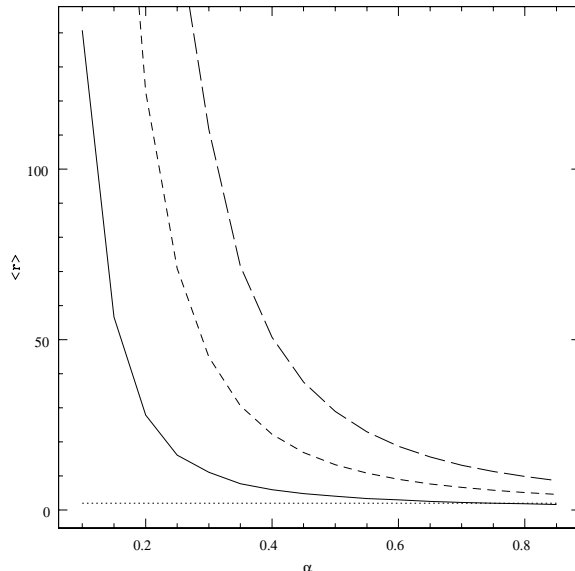


Figure 6: The expectation value of r , in units of GM/c^2 , for the $1S_{1/2}$, $2S_{1/2}$ and $3S_{1/2}$ states. These are the solid, short dashed and long dashed lines respectively. The broken horizontal line is the horizon.

$\alpha = 0.5$, the dips themselves are largely washed out and we obtain the somewhat structureless plots shown in figure 5.

Some additional insight into the nature of the $S_{1/2}$ orbitals is obtained by calculating the expectation value of r . With our current gauge choices this is defined in the obvious manner as

$$\langle r \rangle = \frac{\int_0^\infty dr r (|u_1|^2 + |u_2|^2)}{\int_0^\infty dr (|u_1|^2 + |u_2|^2)}. \quad (69)$$

These are calculated via a straightforward Simpson's rule, and the results for the $1S_{1/2}$, $2S_{1/2}$ and $3S_{1/2}$ orbitals are shown in figure 6. We see that $\langle r \rangle$ decreases as the coupling increases, though in all cases the order of the orbitals remains the same. Once the coupling approaches 1 the expectation value for the $1S_{1/2}$ state approaches the horizon size. For higher α the bulk of the probability density must lie inside the horizon, representing a (short lived) state of a particle which has fallen inside the horizon.

While the low angular momentum orbitals are concentrated near the horizon, the orbitals with larger angular momentum still lie an appreciable distance out. As such, they adopt a form closer to the familiar Hydrogen atom orbitals. A series of such orbitals are shown in figure 7, which shows the first excited mode for κ values of 1, 2, 3 and 4. The coupling is again set to 0.5. As expected, the probability density is concentrated successively further from the hole. By the time we reach $\kappa = 3$ (a classical radius of $x = 33$) the wavefunction returns to the familiar Hydrogen-like form.

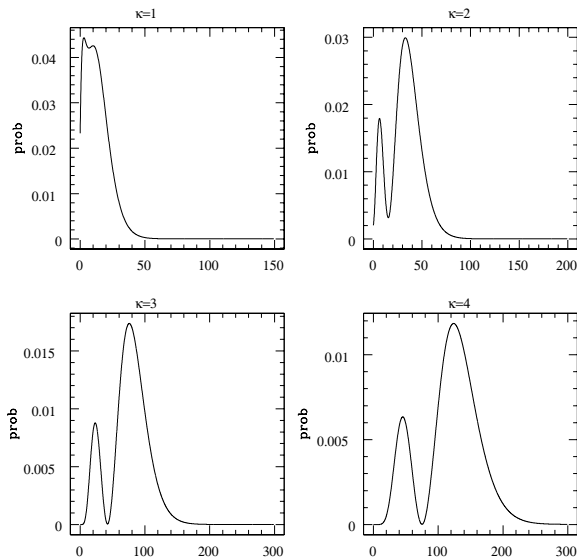


Figure 7: The radial probability density for a range of angular momentum values with a coupling of $\alpha = 0.5$. The first excited states are shown for $\kappa = 1, 2, 3, 4$. As κ increases the orbitals are concentrated further from the source, and begin to resemble Hydrogen atom wavefunctions.

5 Decay rates

So far we have concentrated on the real part of the energy, and the associated orbitals. But the fact that the black hole Hamiltonian is not Hermitian implies that the energy is not real and the states have a finite half-life. As such they should perhaps be more properly referred to as resonance states as opposed to bound states. But for suitably large angular momenta the half lives can be pushed up as high as desired and such states will be extremely long lived. Such states are appropriate for a quantum description of a particle in a classically stable orbit some distance from the horizon.

As argued above, the imaginary part of the energy will be negative, corresponding to a decay. The behaviour of this decay can be visualised in a number of ways. With $E = \omega - i\nu$, the relevant quantity to study is

$$a = \frac{\nu}{mc^2}. \quad (70)$$

In figure 8 we plot a as a function of coupling for the $1S_{1/2}$ state. For comparison the real part of the energy is also plotted. The real energy falls to a minimum and starts increasing again as the orbits become unfavourably close, whereas the imaginary term simply increases monotonically. This as one would expect, as figure (6) showed that the orbits become increasingly tightly bound as α increases. As the coupling strength reaches 1, the imaginary component of the energy is of the order of 0.3 times the rest energy of the particle. This implies that the orbit should decay on the time-scale defined by the Compton

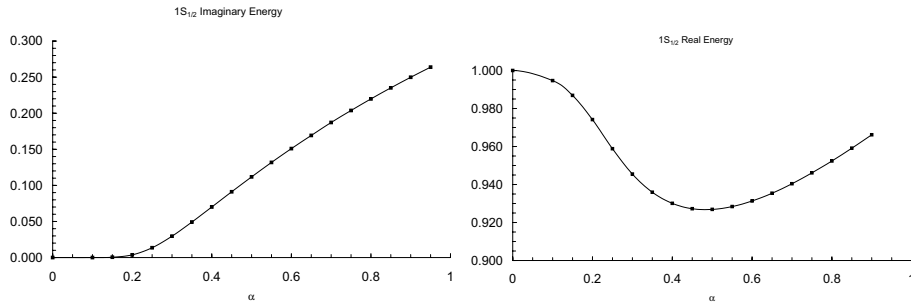


Figure 8: The imaginary and real energies for the $1S_{1/2}$ state. The left hand plot shows (minus) the imaginary component of the energy as a function of the coupling strength. As expected, this increases as the orbits become more tightly bound. For comparison, the more complicated behaviour of the real part of the energy is shown on the right-hand side.

frequency. These states are therefore extremely short lived, with a resonance width comparable to the orbital energy.

In figure 9, a is plotted for the $2S_{1/2}$ and $2P_{3/2}$ states. Both plots show the expected monotonic increase in a with coupling strength as the orbits become more tightly bound and a greater percentage of the wavefunction lies inside the horizon. Comparing with the values for the $1S_{1/2}$ state in figure 8 we see that the decay rates are appreciably smaller, which again is as expected. Increasing the angular momentum further dramatically reduces the decay rate. For example, increasing κ to 3 ($l = 4$) gives an a value of 2.4×10^{-7} for a coupling strength of 0.5. States with higher angular momentum can therefore be extremely stable, as the increase in l keeps the bulk of density away from the singularity.

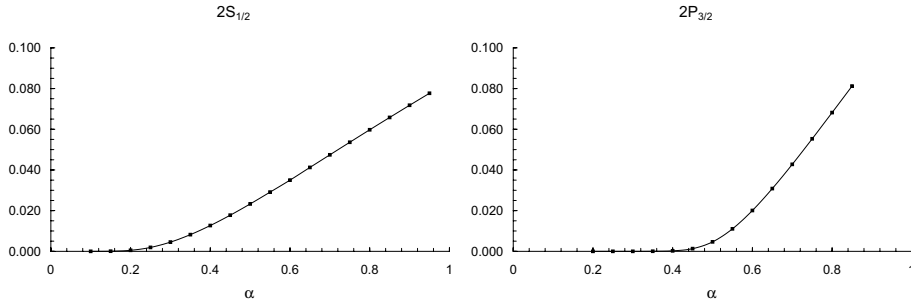


Figure 9: The imaginary energies for the $2S_{1/2}$ (left) and $2P_{3/2}$ (right) states. These increase monotonically with the coupling strength, but are considerably lower than the equivalent $1S_{1/2}$ values.

6 Discussion

We have demonstrated the existence of a complicated spectrum of bound states for a quantum fermion in a black hole background. The qualitative features of the spectrum can be understood in terms of simple semi-classical models, but a full quantitative understanding only seems possible through a mixture of computational methods. The work in this paper can clearly be extended in a number of ways. We have only plotted the spectrum out to a coupling strength of 1, but astrophysical values can be far larger than this. For larger α we expect that the ground state will be one of large angular momentum. In this regime the spectrum will be quite different to that of the Hydrogen atom. One important question is precisely how great a binding energy can be achieved. In figure 2 we see that around $\alpha = 1$ we are achieving total energies of $0.9mc^2$, which are somewhat lower than the classical value of $0.94mc^2$. This suggests that more energy may be available in accretion processes than is traditionally thought. As well as increasing α , it would be of considerable interest to repeat this work for the case of a Kerr black hole. In this respect a useful start has been made in [14], where the Kerr solution is written in a form which generalises the ‘Newtonian’ gauge employed in this paper. The calculations for the Kerr case are more complicated, however, because the angular separation constants are energy-dependent [2, 8].

The energy spectra presented in this paper raise a number of fundamental issues, which demonstrate just how limited our understanding currently is of the interaction of gravity and quantum theory. In quantum electrodynamics we would have little difficulty interpreting energy spectra of the type found here. The energy levels have finite widths, so finite decay times. As the states decay the particles fall into lower energy states and radiate. Presumably similar processes apply to the gravitational problem. The quantum description of a particle falling onto the singularity of a black hole then could involve a series of quantum jumps to lower energy orbits before the wavefunction finally decays and the particle finds itself with an ever increasing probability of being destroyed by the singularity. But this quantum description alters the physics of the process quite dramatically. As the particle undergoes a series of transitions we expect that it should radiate, which does not happen classically. Quite what form this radiation should take (electromagnetic, gravity waves?) is unclear. Also, as a transition takes place we should keep careful track of the evolution of the matter stress-energy tensor to tell us where the radiated energy is concentrated. A related problem this exposes is that we have not considered back reaction on the gravitational field, which could alter this picture. Including back reaction would involve a quantum theory of two particles interacting gravitationally, and such a theory does not yet exist.

The quantum treatment of a particle in a gravitational field exhibits a curious anti-parallelism with the electromagnetic case. In classical electrodynamics a charged particle in orbit around a point source should radiate, making atoms unstable. This problem is resolved by quantum mechanics, which predicts the existence of stable, non-radiating bound states. The reverse is true of gravitation. Classically, a particle can orbit a black hole in a geodesics outside the horizon, and such an orbit is stable. But quantum theory changes this, and states that no totally stable orbits exist, due to the finite probability of the particle finding itself inside the horizon and ending on the singularity. While

the time-scales involved in these decays may be of limited interest astrophysically, such processes are clearly of fundamental importance in understanding the interplay between quantum theory and gravitation.

A final point to raise here is that the spectrum of real energies derived here has a mirror image of negative energy bound states. Each of these negative energy states also has a finite lifetime. If we model the vacuum in terms of a Dirac sea of filled negative energy states, we must include the bound states as well as the free continuum. It then follows that the vacuum itself is decaying — the black hole is sucking in the vacuum. Such a loss of negative energy states is seen as a creation of positive energy modes, which could contribute to Hawking radiation. This contribution appears to have been neglected in previous calculations, which concentrate only on the scattered states [1]. It is well known in calculations of the Lamb shift, for example, that ignoring the bound states in the calculation gives the wrong answer [15]. It would be of great interest to assess the contribution played by bound states to the gravitational analogue of this process.

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